

Supersymmetric Berry Index

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Abstract

We revise the sequences of SUSY for a cyclic adiabatic evolution governed by the supersymmetric quantum mechanical Hamiltonian. The condition (supersymmetric adiabatic evolution) under which the supersymmetric reductions of Berry (nondegenerated case) or Wilczek-Zee (degenerated case) phases of superpartners are taking place is pointed out. The analogue of Witten index (supersymmetric Berry index) is determined. The final expression for new index has compact form of $\text{ind}_B H = \text{sDet } U \equiv \text{Det } U^\tau$, where U is the cyclic evolution operator generated by supersymmetric Hamiltonian H and τ is supersymmetric involution.

As the examples of suggested concept of supersymmetric adiabatic evolution the Holomorphic quantum mechanics on complex plane and Meromorphic quantum mechanics on Riemann surface are considered. The supersymmetric Berry indexes for the models are calculated.

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1 Introduction

During last fifteen years it was proved due to the works by E.Witten [1], A.Jaffe [2], L.Alvarez-Gaume [3] and others that the Supersymmetric Quantum Mechanics (SQM) and Supersymmetric Quantum Field Theory are powerful tools to connect and to splice geometrical and analytical substances. The Witten's approach to the deriving of Morse's inequalities, supersymmetric proof of Index Theorems, the construction of infinitesimal analysis on the base of Supersymmetric Quantum Field Theory, new supersymmetric view on the complex analysis on the Klein surfaces [4] are some examples of such connections. The main technical instrument of the considerations is Witten index of supersymmetric Hamiltonians. It is topologically stable due to spectral properties of supersymmetric systems and only depends on a structure of vacuum subspace of the theory. On the other hand Witten index can be realized as a partition function "twisted" by the supersymmetric involution and can be presented in the form of a functional integral. All these facts allow to calculate Witten index by two ways. The first one is to use operator theory to investigate vacuum subspace of the Hamiltonian. As a result geometrical nature emerges. The second way uses Quantum Field Theory methods to compute functional integrals. This way leads to the analytical substances. It is possible to refer to Chern-Gauss-Bonne Theorem and the theorem for real-meromorphic functions on the Klein surfaces [4] as some results of the realization of the program. All these prompt us to look for new topological indices.

The main ideas of SQM approach to the calculation of new topological indices can be developed in the close analogy with supersymmetric Witten index. We have to construct quantum mechanical quantity in the framework of SQM such that:

1. This quantity is calculated by means of Quantum Field Theory methods;
2. It only depends on vacuum state properties and the contributions of superpartners with nonzero energy are vanishing.

Some of the attempts on this direction are Supersymmetric Scattering Index in the Supersymmetric Scattering Theory [5] and GSQM- indices [6] in Generalized Supersymmetric Quantum Mechanics (GSQM) connected with q - deformation of Extended supersymmetrical Quantum Mechanics [7]. In this paper we introduce new topological supersymmetric index based on the concept of Berry phases.

The discovery of topological phases in cyclic adiabatic evolution [8, 10] immediately generated a flow of papers devoted the subject. It was very natural to splice two topological objects: Berry phases and Witten index. However in the papers [11] it was shown that the phases of superpartners are different in general and hence there is no possibility to invent the topologically stable index. To escape this difficulty we formulate the condition which leads to the vanishing the difference of the phases and observe this phenomenon for some well-known models. This let us to introduce new Supersymmetric topological index – Supersymmetric Berry Index (Index of Cyclic Adiabatic Supersymmetric Evolution, CASE-index). For some partial cases the index is an exponent of the difference of Berry phases of zero-modes in "bosonic" and "fermionic" subspaces. In this feature our index is close to the Supersymmetric Scattering Index which "calculates" the difference of scattering phases [5].

The paper is organized as follows. In the section 2 we remind the notion of topological phases for cyclic adiabatic evolution and introduce convenient notation. In the section 3 we prove some useful propositions for adiabatic evolution of the supersymmetric systems. In section 4 we formulate the conditions of Cyclic Adiabatic Supersymmetric Evolution (CASE) and define the Supersymmetric Berry Index (CASE-index). This index is calculated for the Holomorphic and Meromorphic Supersymmetric Quantum Mechanics in the section 5. In the last section 6 we formulate conclusion remarks on the possibility to use Supersymmetric Berry Index to derive new index theorems.

2 Adiabatic evolution

Let's consider the quantum mechanical system governed by time depended Hamiltonian $H(t)$, $t \in [0, T]$. We assume that for all $t \in [0, T]$ Hamiltonian $H(t)$ has only discrete spectrum. $E_j(t)$ denote its

eigenvalues and $P_j(t)$ denote projectors on corresponding eigenspaces. We assume that $E_j(t)$ and $P_j(t)$ are continuous function of t . Denote $s = t/T$.

Theorem 1 (*Adiabatic theorem*) [12] *If the adiabatic conditions*

$$\begin{aligned} 1^0 \quad & E_j(s) \neq E_k(s) \quad \forall s \in [0, 1], \quad j \neq k, \\ 2^0 \quad & \forall j \quad P_j(s) \text{ is double continuously} \\ & \text{differentiable function of } s \in [0, 1], \end{aligned} \quad (1)$$

take place and evolution operator $U_T(s)$ obeys Schrödinger equation

$$i \frac{\partial}{\partial s} U_T(s) = T H(s) U_T(s)$$

Then

$$\lim_{T \rightarrow \infty} U_T(s) P_j(0) = P_j(s) \lim_{T \rightarrow \infty} U_T(s) \quad (2)$$

Now we use this theorem for the deriving the form of the adiabatic evolution operator in terms of dynamical and geometrical phases. Condition 2^0 provides that $\dim P_j(t) \mathbf{H}$ does not depend on time t . In $P_j(t) \mathbf{H}$ let's choose the basis $\{\varphi_j^\alpha(t)\}_{\alpha=1}^{\dim P_j \mathbf{H}}$ and consider wave function $\psi_j^\alpha(t)$ with initial condition

$$\psi_j^\alpha(0) = \varphi_j^\alpha(0). \quad (3)$$

Adiabatic theorem says that at any moment $t \in [0, T]$ the wave function $\psi_j^\alpha(t)$ in the adiabatic limit is eigenfunction of instant Hamiltonian with eigenvalue $E_j(t)$. Therefore it can be decomposed in the basis:

$$\psi_j^\alpha(t) = \sum_{\alpha'=1}^{\dim P_j \mathbf{H}} u_j^{\alpha' \alpha}(t) \varphi_j^{\alpha'}(t) \quad (4)$$

The substitution of the last expression in Schrödinger equation

$$i \frac{\partial}{\partial t} \psi_j^\alpha(t) = H(t) \psi_j^\alpha(t) \quad (5)$$

gives

$$\sum_{\alpha'=1}^{\dim P_j \mathbf{H}} \left(i \dot{u}_j^{\alpha' \alpha}(t) \varphi_j^{\alpha'}(t) + i u_j^{\alpha' \alpha}(t) \dot{\varphi}_j^{\alpha'}(t) - E_j u_j^{\alpha' \alpha}(t) \varphi_j^{\alpha'}(t) \right) = 0 \quad (6)$$

The scalar product of this equation with $\langle \varphi_j^\beta(t) |$ result in the equation for $u_j^{\beta \alpha}(t)$:

$$i \dot{u}_j^{\beta \alpha}(t) + \sum_{\alpha'} i u_j^{\alpha' \alpha}(t) \langle \varphi_j^\beta(t) | \dot{\varphi}_j^{\alpha'}(t) \rangle - E_j u_j^{\beta \alpha}(t) = 0 \quad (7)$$

Let's introduce matrices

$$U(t) = \|u_j^{\alpha \alpha'}(t)\|, \quad B(t) = \|\langle \varphi_j^\alpha(t) | \dot{\varphi}_j^{\alpha'}(t) \rangle\|, \quad E(t) = \|\delta_j^{\alpha \alpha'} E_j(t)\|, \quad (8)$$

which are block-diagonal in adiabatic limit. Blocks (numerated by j) correspond to energy levels $E_j(t)$ and their dimensions are equal to the degrees of degeneracy of the levels. The equation (7) can be written in the matrix form:

$$\dot{U}(t) = -(B(t) + iE(t))U(t) \quad (9)$$

Taking into account the initial condition $U(0) = I$ we can write its solution:

$$U(t) = \exp \left(-i \int_0^t E(s) ds \right) \text{Texp} \left(- \int_0^t B(s) ds \right) \quad (10)$$

In RHS of the expression the first factor has dynamical nature the second factor is of geometrical one. It is the second term that in the case of cyclic adiabatic evolution gives the geometrical Berry phases.

3 Difference of the superpartners's phases

The main purpose of this section is to introduce supersymmetric notations, describe the difference in Berry phases for eigenstates – superpartners of supersymmetric Hamiltonian $H(t)$ [11] and to prove some statements we use below in section 3.

Let's suppose that the Hamiltonian $H(t)$ which manages the cyclic adiabatic evolution is supersymmetric one i.e. the relations of the supersymmetric quantum mechanics (SQM) take place at any instance of $[0, T]$:

$$\begin{aligned} \tau = \tau^* = \tau^{-1} , \quad & Q^*(t) = Q(t) , \\ \tau Q(t) + Q(t)\tau = 0 , \quad & H(t) = \left(Q(t) \right)^2 \end{aligned} \quad (11)$$

Here τ is supersymmetric involution (grading operator on Hilbert space of physical states) and $Q(t)$ is supercharge for the supersymmetric Hamiltonian $H(t)$. In accordance with τ -grading Hilbert space splits in two subspaces ("bosonic" \mathbf{H}_+ and "fermionic" \mathbf{H}_- spaces) such that

$$\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_- , \quad \tau \mathbf{H}_\pm = \pm \mathbf{H}_\pm \quad (12)$$

On this basis the operators τ , $Q(t)$, $H(t)$ can be rewritten in the matrix form:

$$\begin{aligned} \tau &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad Q(t) = \begin{pmatrix} 0 & q(t) \\ q^*(t) & 0 \end{pmatrix} , \\ H(t) &= \begin{pmatrix} H_+(t) & 0 \\ 0 & H_-(t) \end{pmatrix} = \begin{pmatrix} q(t)q^*(t) & 0 \\ 0 & q^*(t)q(t) \end{pmatrix} , \end{aligned} \quad (13)$$

with $q(t) : D(q(t)) \rightarrow \mathbf{H}_-$ densely defined closed operator on the domain $D(q(t)) \subset \mathbf{H}_+$, where $D(q(t)) = D(Q(t)) \cap \mathbf{H}_+$. Operators $H_+(t)$ and $H_-(t)$ are called the Hamiltonian of superpartners. With the assumption about pure discrete spectrum of the Hamiltonian $H(t)$ the relations (11) provide the coincidence of the spectra of $H_+(t)$ and $H_-(t)$ at any instance excepting zero-energy levels if any.

Generally there is no connection between the zero-modes (eigenfunctions with zero eigenvalue) of H_+ , H_- in \mathbf{H}_+ and \mathbf{H}_- but for nonzero-modes it is possible to do this. However the following lemma take place.

Lemma 1 1. *Witten index of supersymmetric Hamiltonian $H(t)$*

$$\text{ind}_W H(t) = \dim \ker \mathbf{H}_+(t) - \dim \ker \mathbf{H}_-(t)$$

is adiabatic invariant, i.e. in the conditions of adiabatic evolution the following equality takes place:

$$\text{ind}_W H(t) = \text{ind}_W H(0) \quad \forall t \in [0, T].$$

2. *If $\text{ind}_W H(0) \neq 0$ then dimensions of "bosonic" and "fermionic" zero-mode subspaces are also adiabatic invariants, i.e.*

$$\begin{aligned} \dim \ker \mathbf{H}_+(t) &= \dim \ker \mathbf{H}_+(0) , \\ \dim \ker \mathbf{H}_-(t) &= \dim \ker \mathbf{H}_-(0) \quad \forall t \in [0, T] . \end{aligned}$$

Proof. Let $\dim \ker H_+(0) = m$, $\dim \ker H_-(0) = n$. Then under the adiabatic evolution which does not change the degeneracy of instant eigenvalues only m eigenfunctions could leave zero-mode subspace in \mathbf{H}_+ at some moment t_0 . Due to supersymmetry arguments this leads to the fact that m eigenfunctions leave at moment t_0 zero-mode subspace in \mathbf{H}_- (because as it was noted above $\text{spec} H_- \setminus \{0\} = \text{spec} H_+ \setminus \{0\} \forall t \in [0, T]$). So there are two possibilities:

1. $m = n$ and $\text{ind}_W H(0) = \text{ind}_W H(t_0) = 0$. It is possible to investigate the inverse process and to consider the arriving of $2m$ nonzero-modes in zero-mode subspace. This process does not also change Witten index. Summarizing the reflection we infer $\text{ind}_W H(t) = \text{ind}_W H(0)$ for $\forall t \in [0, T]$. This equality proves the Lemma 1 for the case of $\text{ind}_W H(0) = 0$.
2. $m \neq n$ and $\text{ind}_W H(0) = m - n \neq 0$. Then zero-mode subspace in the space \mathbf{H}_- splits at the moment t_0 that contradicts to the adiabaticity of the evolution. So there is no possibility for zero-modes to leave zero-mode subspace in \mathbf{H}_- and hence in \mathbf{H}_+ . This note results in the next equalities:

$$\begin{aligned} \dim \ker \mathbf{H}_+(t) &= \dim \ker \mathbf{H}_+(0) , \\ \dim \ker \mathbf{H}_-(t) &= \dim \ker \mathbf{H}_-(0) , \\ \text{ind}_W H(t) &= m - n = \text{ind}_W H(0) \quad \forall t \in [0, T], \end{aligned} \tag{14}$$

which finish the proof of the Lemma 1 \square

Now let's consider the instant eigenfunctions $\{\varphi_{i+}^\alpha\}_{\alpha=1}^n \in \mathbf{H}_+$, $\{\varphi_{i-}^\alpha\}_{\alpha=1}^n \in \mathbf{H}_-$ corresponded to the instant eigenvalue $E_i(t) \neq 0$ with the degeneracy n . It is possible to choose such basis in \mathbf{H}_+ , \mathbf{H}_- that $\{\varphi_{i+}^\alpha\}$, $\{\varphi_{i-}^\alpha\}$ are expressed one through another:

$$\begin{aligned} q^*(t)\varphi_{i+}^\alpha(t) &= \sqrt{E_i(t)}\varphi_{i-}^\alpha(t) \\ q(t)\varphi_{i-}^\alpha(t) &= \sqrt{E_i(t)}\varphi_{i+}^\alpha(t) . \end{aligned} \tag{15}$$

The eigenfunctions related as (15) are said to be superpartners.

It is interesting to compare the phases gained by wave functions of superpartners under cyclic adiabatic evolution. It is obvious that the dynamical phases are equal. However geometrical phases (Berry phases) can differ. The Theorem 2 shows it.

Theorem 2 [11] *In the notations (8)*

$$\begin{aligned} \Delta_j^{\alpha\alpha'}(t) &\equiv B_{j+}^{\alpha\alpha'}(t) - B_{j-}^{\alpha\alpha'}(t) = \\ &= \frac{\dot{E}_j(t)}{2E_j(t)}\delta^{\alpha\alpha'} + \langle \varphi_{j+}^\alpha(t) | \frac{-q\dot{q}^*}{E_j(t)} | \varphi_{j+}^{\alpha'}(t) \rangle \end{aligned} \tag{16}$$

Proof. The proof of the Theorem 2 can be fulfilled by straightforward calculation using the definition of the matrix B and relations (15). \square

Corollary 2.1 Using the antihermitian property of the matrix $B_{j\pm}$: $B_{j\pm}^{\alpha\alpha'} = -\overline{B_{j\pm}^{\alpha'\alpha}}$ it is possible to rewrite $\Delta_j^{\alpha\alpha'}(t)$ in symmetrical form

$$\Delta_j^{\alpha\alpha'}(t) = \frac{\Delta_j^{\alpha\alpha'}(t) - \overline{\Delta_j^{\alpha'\alpha}(t)}}{2} = \langle \varphi_{j+}^\alpha(t) | \frac{\dot{q}(t)q^*(t) - q(t)\dot{q}(t)^*}{2E_j(t)} | \varphi_{j+}^{\alpha'}(t) \rangle$$

which leads to the expressions $\Delta_j^{\alpha\alpha'}(t)$ through the supercharge $Q(t)$:

$$\begin{aligned} \Delta_j^{\alpha\alpha'}(t) &= \frac{1}{2E_j(t)} \langle \varphi_{j+}^\alpha(t) | [\dot{Q}(t), Q(t)] | \mathbf{H}_+ | \varphi_{j+}^{\alpha'}(t) \rangle = \\ &= -\frac{1}{2E_j(t)} \langle \varphi_{j-}^\alpha(t) | [\dot{Q}(t), Q(t)] | \mathbf{H}_- | \varphi_{j-}^{\alpha'}(t) \rangle \end{aligned}$$

The latter expression can be obtained by the same way starting from eigenfunction in \mathbf{H}_- .

Corollary 2.2 $\text{Tr } \Delta_j$ can be expressed in supersymmetric terms

$$\text{sTr } B_j(t) = \text{Tr } \Delta_j(t) = \frac{1}{4} \text{sTr}(H^{-1}(t)[\dot{Q}(t), Q(t)]P_j(t)) . \tag{17}$$

These relations allows us to introduce the notion of Adiabatic Supersymmetric evolution in the next section.

4 Cyclic Adiabatic Supersymmetric Evolution (CASE) and its index

From here on we assume that adiabatic evolution is cyclic one on $[0, T]$ i.e.

$$H(T) = H(0) \quad (18)$$

In this case we can take instant eigenfunction which obey cyclic condition:

$$\varphi_i^\alpha(T) = \varphi_i^\alpha(0) \quad (19)$$

The main problem of this section is to define some class of supersymmetric Hamiltonians which allow to construct topologically stable index on the basis of cyclic adiabatic phase.

The usual way to calculate topological invariants into the framework of supersymmetry is to invent the quantity in which the contributions of superpartners with nonzero eigenvalues are cancelled and the rest depends on vacuum subspace structure. In this way Witten index can be realized as some trace on the space of states:

$$\text{ind}_W H = \text{Tr} \left(\tau e^{-\beta H} \right) \quad \forall \beta > 0 \quad (20)$$

and its stability is a consequence of the cancelation. We would like to suggest the analog of the construction based on Berry phases. On the other hand Theorem 2 describes the differences for phases of superpartners in general. This compels us to reduce the set of supersymmetrical Hamiltonians to extract the subset of operators for which the difference effectively disappears. At first we describe the subset and introduce new supersymmetric index for it. Then we show that this class of supersymmetrical Hamiltonian is enough wide to contain well-known interesting examples.

Definition 1. The supersymmetrical Hamiltonian $H(t)$ admits the Cyclic Adiabatic Supersymmetric Evolution (CASE) on $[0, T]$ if:

1. $H(t)$ obeys the SQM algebra (11);
2. $H(t)$ governs the adiabatic evolution i.e. adiabatic condition (1) take place;
3. $H(T) = H(0)$;

$$\begin{aligned} 4. \int_0^T s\text{Tr}_{reg}(H^{-1}(t)[\dot{Q}(t), Q(t)])dt &\equiv \\ &\equiv \lim_{\lambda \rightarrow \infty} \int_0^T s\text{Tr}(H^{-1}(t)[\dot{Q}(t), Q(t)]E([0, \lambda])(t))dt = 0 \end{aligned} \quad (21)$$

where $E([0, \lambda])(t)$ is the spectral measure of the interval $]0, \lambda]$ for the operator $H(t)$.

For the class of CASE-Hamiltonians it is possible to introduce an analog of supersymmetric Witten index and index of Supersymmetric Scattering Theory [5]. It is Supersymmetric Berry Index.

Definition 2. Let's $H(t)$ is CASE-Hamiltonian on $[0, T]$. Then its Supersymmetric Berry Index is defined by the following relation:

$$\text{ind}_B H = s\text{Det}_{reg} U(T) \equiv \lim_{\lambda \rightarrow \infty} \text{Det} U^\tau(T) \Big|_{E([0, \lambda])(t) \mathbf{H}} \quad (22)$$

where

$$U^\tau(t) \equiv \begin{pmatrix} U_+(t) & 0 \\ 0 & U_-^{-1}(t) \end{pmatrix}$$

and $U_{\pm}(t)$ are evolution operators for the Hamiltonians $H_{\pm}(t)$:

$$U_{\pm}(t) = \text{Texp}(-i \int_0^t H_{\pm}(t) dt)$$

Now we prove that the eigenfunctions with nonzero eigenvalues does not contribute in $\text{ind}_B H$ and calculate Berry phases of zero-modes. The following theorem formalizes the statement:

Theorem 3 [11] *Let's $H(t)$ is CASE-Hamiltonian on $[0, T]$ and $\text{ind}_W H(0) \neq 0$ then*

$$\text{ind}_B H \equiv \text{sDet}_{\text{reg}} U(T) = \exp \left(- \int_0^T \text{sTr}(B(t) P_0(t)) dt \right) \quad (23)$$

where

$$B(t) \equiv \begin{pmatrix} B_+(t) & 0 \\ 0 & B_-(t) \end{pmatrix}$$

and $P_0(t)$ is the projector on $\ker H(t)$.

Proof. At first we prove that RHS is well-defined and independent on the choice of instant bases in $\ker H_{\pm}(t)$ which keep their dimensions due to Lemma 1.

Lemma 2 $\exp(-\int_0^T \text{sTr}(B(t) P_0(t)) dt)$ does not depend on the choice of instant orthonormalized bases $\{\varphi_{0\pm}^{\alpha}(t)\}$ with cyclic condition (19) in $\ker H_{\pm}(t)$.

Proof. If we introduce new bases $\{\chi_{0\pm}^{\alpha}(t)\}$ in $\ker H_{\pm}(t)$ by the relations:

$$\chi_{0\pm}^{\alpha}(t) = \sum_{\alpha'=1}^{\dim \ker \mathbf{H}_{\pm}} v_{\pm}^{\alpha\alpha'}(t) \varphi_{0\pm}^{\alpha'}(t) \quad (24)$$

then $\text{Tr} B(t)|_{\ker H_{\pm}(t)}$ can be calculated in new bases:

$$\begin{aligned} \text{Tr}(B(t)|_{\ker H_{\pm}(t)})_{\chi} &= \sum_{\alpha=1}^{\dim \ker \mathbf{H}_{\pm}} \langle \chi_{0\pm}^{\alpha}(t) | \dot{\chi}_{0\pm}^{\alpha}(t) \rangle = \\ &= \sum_{\alpha, \beta, \gamma=1}^{\dim \ker \mathbf{H}_{\pm}} \langle v_{\pm}^{\alpha\beta}(t) \varphi_{0\pm}^{\beta}(t) | v_{\pm}^{\alpha\gamma}(t) \dot{\varphi}_{0\pm}^{\gamma}(t) + \dot{v}_{\pm}^{\alpha\gamma}(t) \varphi_{0\pm}^{\gamma}(t) \rangle = \\ &= \sum_{\alpha, \beta, \gamma=1}^{\dim \ker \mathbf{H}_{\pm}} (\langle \varphi_{0\pm}^{\beta}(t) | \dot{\varphi}_{0\pm}^{\gamma}(t) \rangle \delta^{\beta\gamma} + \delta^{\beta\gamma} v_{\pm}^{-1\beta\alpha} \dot{v}_{\pm}^{\alpha\gamma}) = \\ &= \text{Tr}(B(t)|_{\ker H_{\pm}(t)})_{\varphi} + \frac{\partial}{\partial t} \ln \text{Det} \|v_{\pm}^{\alpha\beta}(t)\| \end{aligned} \quad (25)$$

where we use the formula

$$\frac{\partial}{\partial t} \text{Det} V(t) = \text{Det} V(t) \text{Tr} \left(\frac{\partial V(t)}{\partial t} V^{-1}(t) \right) \quad (26)$$

Hence

$$\text{sTr}(B(t)|_{\ker H(t)})_{\chi} = \text{sTr}(B(t)|_{\ker H(t)})_{\varphi} + \frac{\partial}{\partial t} \ln \frac{\text{Det} \|v_+^{\alpha\beta}(t)\|}{\text{Det} \|v_-^{\alpha\beta}(t)\|} \quad (27)$$

$\|v_{\pm}^{\alpha\beta}(T)\|$ is equal to $\|v_{\pm}^{\alpha\beta}(0)\|$ due to the bases $\{\varphi_{0\pm}^{\alpha}(t)\}$ and $\{\chi_{0\pm}^{\alpha}(t)\}$ obey the cyclic condition (19). Therefore last term after the integration by t from 0 to T gives $2\pi i k$, $k \in \mathbb{Z}$ due to the logarithm of complex number is multivaluable function. \square

Now let's return to the proof of the Theorem 3. Keeping in mind the regularization we can calculate the time derivative $\frac{\partial}{\partial t} \text{Det } U^\tau(t)$:

$$\begin{aligned} \frac{\partial}{\partial t} \text{Det } U^\tau(t) &= \frac{\partial}{\partial t} (\text{Det } U_+(t) \text{Det } U_-^{-1}(t)) = \\ &= \frac{\partial}{\partial t} \text{Det } U_+(t) (\text{Det } U_-(t))^{-1} + \text{Det } U_+(t) \frac{\partial}{\partial t} (\text{Det } U_-(t))^{-1} \end{aligned} \quad (28)$$

using formulae (26) and (9) for the systems governed by Hamiltonians $H_\pm(t)$ we get

$$\begin{aligned} \frac{\partial}{\partial t} \text{Det } U^\tau(t) &= \\ &= \text{Det } U^\tau(t) \left(-\text{Tr}(B_+(t) + iE_+(t)) + \text{Tr}(B_-(t) + iE_-(t)) \right) = \\ &= -\text{Det } U^\tau(t) \text{sTr } B(t) \end{aligned} \quad (29)$$

We can solve this equation. Taking into account the initial condition $\text{Det } U^\tau(0) = 1$:

$$\text{Det } U^\tau(T) \Big|_{E([0,\lambda])(t)\mathbf{H}} = \exp\left(-\int_0^T \text{sTr } B(t) \Big|_{E([0,\lambda])(t)\mathbf{H}} dt\right) \quad (30)$$

Due to CASE-condition (21) RHS of this equation has $\lambda \rightarrow \infty$ limit. Therefore

$$\text{sDet}_{reg} U(T) = \text{Det } U^\tau(T) \Big|_{P_0(T)\mathbf{H}} = \exp\left(-\int_0^T \text{sTr } B(t) \Big|_{P_0(t)\mathbf{H}} dt\right) \quad (31)$$

Thus we have proved Theorem 3 \square

Note 3.1 It is possible to generalize Theorem 3 on the case of CASE-Hamiltonians with $\text{ind}_W H(t) = 0$. Then in the statement we have to replace $\exp(-\int_0^T (B(t)P_0(t))dt)$ by $\exp(-\int_0^T (B(t)\tilde{P}_0(t))dt)$ where $\tilde{P}_0(t)$ is projector on eigenspace with eigenvalue which somewhere on interval $[0, T]$ comes in zero.

Note 3.2 For the case of 1-dimensional $\ker \mathbf{H}_\pm$ index $\text{ind}_B H$ gives us no more than $\exp(\Delta\varphi)$ where $\Delta\varphi$ is a difference of Berry phases of zero-modes in "bosonic" and "fermionic" spaces. In this form ind_B is analog of supersymmetric scattering index [5] which also calculates the difference of phases (scattering phases) in "bosonic" and "fermionic" spaces.

In general case the Supersymmetric Berry Index is a complex number on the unit circle and this number can be changed via the variation of the CASE-Hamiltonian. It would be especially interesting to describe some situations for which this number has to be in the set of discrete numbers. Now we formulate the simplest condition on the CASE-Hamiltonian which leads to the discreteness of possible Supersymmetric Berry Indices.

Let's Hilbert space \mathbf{H} has additional structure which we will call "conjugation". Formally it means that there is the involution $P : \mathbf{H} \rightarrow \mathbf{H}$ such that for $\forall \varphi, \psi \in \mathbf{H}$ and $c \in \mathbb{C}$

1. $P^2 = I$
2. $Pc\varphi = \bar{c}P\varphi$
3. $\langle P\varphi | P\psi \rangle = \overline{\langle \varphi | \psi \rangle}$

Theorem 4 [11] *If the "conjugation" P is consistent with the supersymmetric involution i.e. $[P, \tau] = 0$ and CASE-Hamiltonian obeys the condition:*

$$H(t)P = PH(t) \quad \forall t \in [0, T]$$

then

$$\text{ind}_B H(t) = \pm 1$$

Note 4.1 *In this form the Theorem 4 generalize well-known fact [9] that Berry phase of real Hamiltonian is equal zero.*

Proof. From the facts that the "conjugation" P commutes with the Hamiltonian and $P\mathbf{H}_\pm = \mathbf{H}_\pm$ immediately follows that for $\forall t \in [0, T]$ it is possible to choose "real" instant eigenfunctions of the operators $H_\pm(t)$ i.e. such that $P|\chi^\alpha(t)\rangle = |\chi^\alpha(t)\rangle$. We also suppose the orthonormalization condition for the function. It leads to the next property of the instant eigenfunctions:

$$\langle \chi^\alpha(t) | \chi^\beta(t') \rangle = \langle P\chi^\alpha(t) | P\chi^\beta(t') \rangle = \overline{\langle \chi^\alpha(t) | \chi^\beta(t') \rangle} \in R$$

and therefore $\langle \chi^\alpha(t) | \dot{\chi}^\beta(t') \rangle \in R$ for all moments $t, t' \in [0, T]$.

However in general case $\chi_\alpha(T) \neq \chi_\alpha(0)$ Using formulae (23,27) we get following expression for $\text{ind}_B H$:

$$\text{ind}_B H = \exp \left(\int_0^T \sum_{\pm} \pm \sum_{\alpha=1}^{\dim \ker \mathbf{H}_\pm} \langle \chi_{0\pm}^\alpha(t) | \dot{\chi}_{0\pm}^\alpha(t) \rangle dt + \ln \frac{\text{Det} \|v_+^{\alpha\beta}(t)\|}{\text{Det} \|v_-^{\alpha\beta}(t)\|} \Big|_0^T \right)$$

where $v_\pm^{\alpha\beta}(t)$ is the transfer matrix from basis $\{\chi_{0\pm}^\alpha(t)\}$ to basis $\{\varphi_{0\pm}^\beta(t)\}$ in $\ker H_\pm(t)$. $\langle \chi_{0\pm}^\alpha(t) | \dot{\chi}_{0\pm}^\alpha(t) \rangle = 0$ because on the one hand it is real but on another hand it is imaginary due to normalization condition on $\chi_{0\pm}^\alpha(t)$. So

$$\text{ind}_B H = \frac{\text{Det}(\|v_+^{\alpha\beta}(0)\|^{-1} \|v_+^{\beta\gamma}(T)\|)}{\text{Det}(\|v_-^{\alpha\beta}(0)\|^{-1} \|v_-^{\beta\gamma}(T)\|)} \quad (32)$$

Due to the cyclic condition (19) the matrices $\|v_\pm(0)\|^{-1} \|v_\pm(T)\|$ are the transfer matrices from $\{\chi_{0\pm}^\alpha(T)\}$ to $\{\chi_{0\pm}^\alpha(0)\}$:

$$\|v_\pm(0)\|^{-1} \|v_\pm(T)\|^{\alpha\beta} = \langle \chi_{0\pm}^\beta(T) | \chi_{0\pm}^\alpha(0) \rangle \in R$$

Matrices $\|v_\pm(0)\|^{-1} \|v_\pm(T)\|$ are unitary and real. Therefore both determinant in (32) are equal to ± 1 and $\text{ind}_B H = \pm 1$. \square

5 Examples

It is well-known how important to find the simple example which illustrates the general structure and is not shaded the treatment by long calculations. As such example we can consider the supersymmetric harmonic oscillator on complex plane.

5.1 Supersymmetric harmonic oscillator on complex plane.

The supersymmetric harmonic oscillator on a complex plane is the simplest example of the suggested in [2] Holomorphic Supersymmetric Quantum Mechanics which was investigated in many papers [13, 14, 15]. It was shown that the Hamiltonian of supersymmetric harmonic oscillator has pure point spectrum and there is the only single zero mode in "fermionic" subspace. So Witten index for such operator is equal to -1 . Now we demonstrate that this Hamiltonian can be regarded as CASE-one and calculate the Supersymmetric Berry Index. For this supersymmetric system the supercharge $Q(t)$ has the form:

$$Q(t) = \sqrt{2} \begin{pmatrix} 0 & 0 & \bar{c}(t)\bar{z} & -\frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial \bar{z}} & -c(t)z \\ c(t)z & -\frac{\partial}{\partial \bar{z}} & 0 & 0 \\ \frac{\partial}{\partial z} & -\bar{c}(t)\bar{z} & 0 & 0 \end{pmatrix},$$

The corresponding supersymmetric Hamiltonian $H(t) = Q^2(t)$ can be put down:

$$H(t) = \begin{pmatrix} H_+(t) & 0 \\ 0 & H_-(t) \end{pmatrix},$$

where

$$H_+(t) = 2 \begin{pmatrix} |c(t)z|^2 - \frac{\partial^2}{\partial z \partial \bar{z}} & 0 \\ 0 & |c(t)z|^2 - \frac{\partial^2}{\partial z \partial \bar{z}} \end{pmatrix},$$

$$H_-(t) = 2 \begin{pmatrix} |c(t)z|^2 - \frac{\partial^2}{\partial z \partial \bar{z}} & c(t) \\ \bar{c}(t) & |c(t)z|^2 - \frac{\partial^2}{\partial z \partial \bar{z}} \end{pmatrix},$$

and together with supersymmetric involution $\tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ forms the SQM-algebra. In the presented formulae time dependence appears through the arbitrary smooth function $c(t)$ such that $c(0) = c(T)$ and $c(t) \neq 0$ for $\forall t \in [0, T]$.

Lemma 3 $H(t)$ is CASE-Hamiltonian

Proof. Indeed, it is sufficient to write down $[\dot{Q}(t), Q(t)] \big|_{\mathbf{H}_+}$:

$$\begin{aligned} & [\dot{Q}(t), Q(t)] \big|_{\mathbf{H}_+} = \\ & = 2 \begin{pmatrix} \overline{\dot{c}(t)}c(t)|z|^2 - \overline{c(t)}\dot{c}(t)|z|^2 & -2\overline{\dot{c}(t)}z\frac{\partial}{\partial \bar{z}} \\ -2\dot{c}(t)z\frac{\partial}{\partial \bar{z}} & \overline{c(t)}\dot{c}(t)|z|^2 - \dot{c}(t)c(t)|z|^2 \end{pmatrix} \end{aligned} \quad (33)$$

and to take eigenfunctions of $H_+(t)$ in the form

$$\varphi_{j+}^{2\alpha-1} = \begin{pmatrix} \xi_j(t) \\ 0 \end{pmatrix}, \quad \varphi_{j+}^{2\alpha} = \begin{pmatrix} 0 \\ \xi_j(t) \end{pmatrix}.$$

The diagonal terms of the matrix $[\dot{Q}(t), Q(t)] \big|_{\mathbf{H}_+}$ has the opposite signs. Therefore

$$\sum_{\alpha} \langle \varphi_{j+}^{\alpha} | [\dot{Q}(t), Q(t)] | \varphi_{j+}^{\alpha} \rangle = 0 \quad \forall j \text{ such that } E_j(t) > 0$$

This leads to the required equality (21) for the trace. \square

Proposition 1

$$\text{ind}_B H(t) = (-1)^{\text{ind}_{[0,T]} c(t)}$$

Proof. Due to the fact that $H(t)$ is CASE-Hamiltonian with $\text{ind}_W H(t) = -1$ and single zero mode is in "fermionic" sector [2] we can apply the Theorem 3 for calculation of $\text{ind}_B H(t)$.

Using the notation $c(t) = r(t) \exp(i\theta(t))$ and the explicit form of the zero-mode

$$\varphi_{0-}(t) = \sqrt{\frac{r(t)}{\pi}} \exp(-r(t)|z|^2) \begin{pmatrix} \exp(i\theta(t)) \\ -1 \end{pmatrix}$$

we compute $\text{sTr}(B(t)P_0)$:

$$\begin{aligned} & -\text{sTr} B(t) \big|_{\ker H} = \langle \varphi_{0-}(t) | \dot{\varphi}_{0-}(t) \rangle = \\ & = \frac{r(t)}{\pi} \int_C \exp(-2r(t)|z|^2) (i\dot{\theta}(t) + \frac{\dot{r}(t)}{r(t)} - 2\dot{r}(t)|z|^2) d\bar{z}dz = \frac{i}{2} \dot{\theta}(t) \end{aligned} \quad (34)$$

According to the Theorem 3 Supersymmetric Berry Index is equal to

$$\text{ind}_B H = \exp \left(\frac{i}{2} \int_0^T \dot{\theta}(t) dt \right) = (-1)^{\text{ind}_{[0,T]} c(t)} \quad \square$$

This complete the consideration of Supersymmetric Harmonic Oscillator on complex plane.

Now we consider the general case of the Supersymmetric Meromorphic Quantum Mechanics on the Riemann surface. This system contains the previous example as a particular case.

5.2 Meromorphic Supersymmetric Quantum Mechanics on Riemann surface

Let's define SQM on the arbitrary genus compact Riemann surface M_0 with meromorphic superpotential with poles in $z_1, \dots, z_n \in M_0$. To do this Kähler metric g which is euclidean at infinity in the points z_1, \dots, z_n was introduced [16].

For this metric there are open neighborhoods O_{R_i} of z_i and diffeomorphic maps ϕ_i of $O_{R_i} \setminus \{z_i\}$ to open sets $CB_{R_i} = \{u \in C : |u| > R_i\}$ on complex plane such that on each O_{R_i} the metric is the pullback by ϕ_i of the euclidean metric on CB_{R_i} .

Hilbert space is that of square integrable differential forms: $\mathbf{H} \equiv \Lambda_2(M)$ with scalar product

$$\langle \omega | \phi \rangle = \int_M \bar{\omega} \wedge * \phi \quad (35)$$

where $*$ is Hodge operator.

We will consider the time-depended meromorphic superpotential $F(z, t)$ such that its poles $z_1, \dots, z_n \in M_0$ are independent on t and $F(z, T) = F(z, 0)$.

Supercharges, the supersymmetric involution and the Hamiltonian were defined as the closure in \mathbf{H} of correspondent operators defined on $C_0^\infty(M)$ - forms by formulae:

$$\begin{aligned} Q_+(t) &= \frac{\partial}{\partial \bar{z}} d\bar{z} \wedge + F_z(z, t) dz \wedge, & Q_-(t) &= (Q_+(t))^* , \\ \tau &= (-1)^N, & Q(t) &= Q_+(t) + Q_-(t), & H(t) &= (Q(t))^2 \end{aligned} \quad (36)$$

where N is the degree of the form. The index z stands for derivative on z . These operators obey SQM relations (11). In the work [16] it was shown that operator $H(t)$ has compact resolvent and hence pure discrete spectrum.

Let's choose the basis of differential forms: $1, g d\bar{z} \wedge dz/2, dz, d\bar{z}$. The first two forms belong to \mathbf{H}_+ , the two latter forms belong to \mathbf{H}_- . In this basis all operators can be represented in matrix form (for the sake of simplicity we omit arguments z, t of function F):

$$Q(t) = \begin{pmatrix} 0 & 0 & \frac{2}{g} \bar{F}_z & -\frac{2}{g} \frac{\partial}{\partial z} \\ 0 & 0 & \frac{2}{g} \frac{\partial}{\partial \bar{z}} & -\frac{2}{g} F_z \\ F_z & -\frac{\partial}{\partial z} & 0 & 0 \\ \frac{\partial}{\partial \bar{z}} & -\bar{F}_z & 0 & 0 \end{pmatrix}, \quad (37)$$

$$H_+(t) = \begin{pmatrix} \frac{2}{g} (|F_z|^2 - \frac{\partial^2}{\partial z \partial \bar{z}}) & 0 \\ 0 & \frac{2}{g} (|F_z|^2 - \frac{\partial^2}{\partial z \partial \bar{z}}) \end{pmatrix}, \quad (38)$$

$$H_-(t) = \begin{pmatrix} -\frac{\partial}{\partial z} \frac{2}{g} \frac{\partial}{\partial \bar{z}} + \frac{2}{g} |F_z|^2 & \frac{2}{g} F_{zz} \\ \frac{2}{g} \bar{F}_{z\bar{z}} & -\frac{\partial}{\partial \bar{z}} \frac{2}{g} \frac{\partial}{\partial z} + \frac{2}{g} |F_z|^2 \end{pmatrix}.$$

and

$$[\dot{Q}(t), Q(t)] \Big|_{\mathbf{H}_+} = \frac{2}{g} \begin{pmatrix} \dot{\bar{F}}_z F_z - \bar{F}_z \dot{F}_z & -2\dot{\bar{F}}_z \frac{\partial}{\partial z} \\ -2\dot{F}_z \frac{\partial}{\partial \bar{z}} & \bar{F}_z \dot{F}_z - \bar{F}_z F_z \end{pmatrix}. \quad (39)$$

In the subspace \mathbf{H}_+ we can take eigenfunction in the form of

$$\varphi_{j+}^{2\alpha-1}(t) = \begin{pmatrix} \xi_j^\alpha(t) \\ 0 \end{pmatrix}, \quad \varphi_{j+}^{2\alpha}(t) = \begin{pmatrix} 0 \\ \xi_j^\alpha(t) \end{pmatrix},$$

and diagonal terms of matrix $[\dot{Q}(t), Q(t)]|_{\mathbf{H}_+}$ are opposite in the sign therefore the condition (21) holds true in this case due to the same reason as in the previous subsection .

If $F(z, t)$ depends on t such that $H(t)$ obeys adiabatic condition (1) then the operator $H(t)$ is CASE-Hamiltonian. Therefore $\text{ind}_B H$ exists. One can show that the Hamiltonian in question satisfies the conditions of the Theorem 4 (for usual complex conjugation as the involution P) and hence $\text{ind}_B H$ is equal to ± 1 .

To escape the technical difficulties we calculate the index for particular case of $F(z, t) = \exp(i\theta(t))f(z)$, $\theta(T) = \theta(0) + 2\pi L$, $L \in \mathbb{Z}$.

Proposition 2

Let $\chi(M_0)$ is Euler characteristics of compact Riemann surface M_0 and D is a divisor of poles of the differential $F_z dz$, then

$$\text{ind}_B H = (-1)^{L(\chi(M_0) + \deg D)}$$

Proof. According to the Theorem 3 we have to investigate zero-modes. In the work [16] number of zero-modes of Hamiltonians $H_{\pm}(t)$ has been calculated and it was shown that the Hamiltonian $H_+(t)$ has no zero-modes, $H_-(t)$ has $K \equiv \chi(M_0) + \deg D$ zero-modes.

Given the basis of subspace of zero-mode at moment $t = 0$

$$\varphi_{0-}^{\alpha}(0) = \begin{pmatrix} \xi_1^{\alpha} \\ \xi_2^{\alpha} \end{pmatrix}$$

we can construct instant bases for any $t \in [0, T]$:

$$\varphi_{0-}^{\alpha}(t) = \begin{pmatrix} \exp(i\theta(t))\xi_1^{\alpha} \\ \xi_2^{\alpha} \end{pmatrix}$$

$$\text{ind}_B H = \exp \left(\int_0^T \sum_{\alpha=1}^K \langle \varphi_{0-}^{\alpha}(t) | \dot{\varphi}_{0-}^{\alpha}(t) \rangle dt \right) = \int_0^T i\dot{\theta}(t) dt \sum_{\alpha=1}^K \|\xi_1^{\alpha}\|^2 \quad (40)$$

At the moment $t = 0$ we can take real eigenbasis. Then $\|\xi_1^{\alpha}\|^2 = \|\xi_2^{\alpha}\|^2 = 1/2$. Therefore $\text{ind}_B H = (-1)^{L(\chi(M_0) + \deg D)}$ \square

The Proposition 2 generalizes the Proposition 1 of the previous section on the case of the Meromorphic Supersymmetric Quantum Mechanics.

6 Conclusion remarks

In this paper we investigated the possibility to insert the concept of topological phases of cyclic adiabatic evolution (Berry phases) to the framework of Supersymmetric Quantum Mechanics and introduced on this basis new topological index – Supersymmetric Berry Index. To illustrate the scheme this index was calculated for the Holomorphic and Meromorphic Supersymmetric Quantum Mechanics. For these cases index is connected with winding number of parametric function which generates the adiabatic evolution.

However at the end of the paper we would like to outline some directions of developments.

We discussed in the paper only one way to calculation the index - through the explicit computing of the contributions of zero-modes. The second step is to generalize the Index Theorems the sense discussed in the Introduction, namely is to represent the CASE-index in functional integral form that allow us to treat Supersymmetric Berry phase using quantum field theory methods.

The Generalized Supersymmetry has proved that it is additional powerful tool of mathematical investigation into the framework of supersymmetry. So to our mind it is interesting to generalize the treatment of Supersymmetric Berry Phase to case of Generalized Supersymmetry.

The last point is the looking for other possibilities of the discreteness of the index because this is a straightforward way to the real topological stability. For instance it is interesting to find the conditions under which the index takes the value in the set of complex roots of unit.

We are going to return to these questions in the next papers.

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